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# On the Cartan Invariants of a Chevalley Group over $GF(p^n)$

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## INTRODUCTION

There are various approaches to the calculation of characteristic $p$ Cartan invariants of a finite Chevalley group. One question that can be asked is: How do these Cartan invariants vary as the size of the field of definition varies? It is the aim of the present paper to show that the Cartan invariants are sums of traces of products of $n$ matrices, where the field of definition has $p^n$ elements. The matrices themselves are calculated by certain relations among formal characters of the corresponding algebraic group. At first glance the matrices seem large and complicated, but in practice most of the terms are 0. The calculations, at least in low rank cases, are quite manageable and are considerably shorter than those in other approaches, e.g., [3, 5]. We give an example by applying this approach to calculate the first Cartan invariant for $SL(3, p^n)$ and $SU(3, p^n)$.

## 1. PRELIMINARIES AND NOTATION

Let $G$ be a semi-simple simply connected algebraic group defined over $K$, the algebraic closure of a finite field of characteristic $p$. Let $\sigma$ be an endomorphism of $G$ such that the set of fixed points $G_\sigma$ is finite. (We are usually thinking of $\sigma$ as the Frobenius endomorphism.) Let $\rho$ be an automorphism of the Dynkin diagram of $G$. Let $\tau = \tau_{n,\rho} = \sigma^n \rho$. We wish to examine how the representation theory of $KG_\tau$ varies with $n$ and $\rho$.

Let $X$ be the weight lattice of $G$ and let $X^+$ be the set of dominant weights. For $\lambda$ in $X^+$ let $M_\lambda$ be the $KG$ module with highest weight $\lambda$. The endomorphism $\sigma$ induces actions on the set of $KG$ modules and on $X$. We also denote these actions by $\sigma$ and write them exponentially.

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Let $\delta$ be half the sum of the positive roots. Let $X_\sigma$ be the subset of $X^+$ consisting of those weights whose coordinates in terms of the fundamental dominant weights are all equal to or less than those of $\delta^\sigma - \delta$. These form a set of representatives of the residue classes of $X$ modulo $X^\sigma$. For example, when $\sigma$ is the Frobenius map, $X_\sigma$ is the restricted weight region, usually denoted $X_p$.

If $\lambda$ is in $X^+$ then we can write $\lambda = \sum (\lambda(i))^{\sigma^i}$ with $\lambda(i)$ in $X_\sigma$. Then $M_\lambda$ is isomorphic to $\bigotimes_{i=0}^{n-1} (M_{\lambda(i)})^{\sigma^i}$. The restrictions of the $M_\lambda$, $\lambda$ in $X_\sigma$, to $G_\sigma$ give all the irreducible $KG_\sigma$ modules up to isomorphisms.

For $\lambda$ in $X^+$ let $\chi_p(\lambda)$ be the formal character of $M_\lambda$. This is an element of $\mathbb{Z}[X]^W$, the set of elements of the group ring $\mathbb{Z}[X]$ which are invariant under the Weyl group $W$. The set $\{\chi_p(\lambda) : \lambda \in X^+\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}[X]^W$. The elements of $\mathbb{Z}[X]^W$ can be considered as elements of $R(G_\tau)$, the ring of class functions on the $p'$-elements of $G_\tau$. We shall not make a distinction in notation between an element of $\mathbb{Z}[X]^W$ and its restriction to $R(G_\tau)$. The set $\{\chi_p(\lambda) : \lambda \in X_\tau\}$ forms a $\mathbb{Z}$-basis for $R(G_\tau)$. If $\mu$ is in $X_\tau$ and $\lambda$ is in $X^+$ we have $\chi_p(\lambda^\tau + \mu) = \chi_p(\lambda)^\tau \chi_p(\mu)$. The character $\chi_p(\delta^\tau - \delta)$ will be denoted by $st_\tau$ and we will simply write $st$ for $st_\sigma$.

For $\lambda$ in $X_\tau$ let $\Phi_{\lambda,\tau}$ be the character of the projective cover of $M_\lambda$ considered as a $KG_\tau$ module. The set $\{\Phi_{\lambda,\tau} : \lambda \in X_\tau\}$ generates a subring $P(G_\tau)$ of $R(G_\tau)$. Every element of $P(G_\tau)$ is divisible in $R(G_\tau)$ by $St_\tau$.

For $\lambda$ in $X$ let $\chi(\lambda) = \sum_{w \in W} (\operatorname{sgn} w) e(w(\lambda + \delta)) / \sum_{w \in W} (\operatorname{sgn} w) e(w\delta)$. If $\lambda \in X^+$ this is the character of the Weyl module with highest weight $\lambda$. Let $s(\lambda)$ be $\sum_{\lambda'} e(\lambda')$, the sum being over the distinct elements in the $W$-orbit of $\lambda$. By the expression $\operatorname{Mult}(s(\lambda)^\tau \chi_p(\mu), \xi)$ where $\xi$ is in $\mathbb{Z}[X]^W$ we mean the coefficient of $s(\lambda)^\tau \chi_p(\mu)$ when $\xi$ is expressed in terms of the basis, $\{s(\lambda)^\tau \chi_p(\mu) : \mu \in X_\sigma,\ \lambda \in X^+\}$ of $\mathbb{Z}[X]^W$.

## 2. Statement of the Theorem

Let $P_\tau(G)$ be the principal ideal of $\mathbb{Z}[X]^W$ generated by $st_\tau$. The basic idea is to "approximate" the elements $\Phi_{\lambda,\tau}$ by the restriction to $P(G_\tau)$ of certain elements of $P_\tau(G)$ which we shall call $\Psi_{\lambda,\tau}$. These elements were defined in [4] and denoted $\Psi_{\lambda,n}$ there. They are characterized by the following property.

Write $\Psi_{\lambda,\tau} = st_\tau \cdot \psi_{\lambda,\tau}$ where $\psi_{\lambda,\tau} \in \mathbb{Z}[X]^W$. Then for $\lambda, \mu$ in $X_\tau$, $\nu$ in $X^+$,

$$\operatorname{Mult}(s(\nu)^\tau st_\tau, \bar{\psi}_{\lambda,\tau} \chi_p(\mu)) = 0$$

unless $\nu = 0$ and $\mu = \lambda$ when the multiplicity is one.

The multiplicity here is with respect to the basis $\{s(\nu)^\tau \chi_p(\mu) : \nu \in X,\ \mu \in X^+\}$, and the bar denotes complex conjugation.

The $\Psi_{\lambda,\tau}$ can also be defined as the characters of the projective indecomposable modules in Jantzen's category of $u_n - T$ modules. It is shown in [1, Lemma 9] that they have the "tensor product" property that $\Psi_{\lambda,\tau} = \prod_{i=0}^{n-1} (\Psi_{\lambda,\sigma})^{\sigma^i}$. Moreover the highest weight of $\Psi_{\lambda,\tau}$ is $\delta^\tau - \delta - \lambda$.

A formula for $\psi_{\lambda,\sigma}$ is given in [1, Lemma 2]. The formula expresses $\psi_{\lambda,\sigma}$ in terms of the elements $s(\lambda)$. To compute the coefficients one needs to know the decomposition of a Weyl module into irreducibles, as well as certain other quantities whose computation is explained in [2]. One can then use the tensor product property to compute $\psi_{\lambda,\tau}$ for general $\tau$. We shall then assume that characters are known and give a formula for $\Phi_{\lambda,\tau}$ in terms of them.

We now state our main results. For $\xi \in \mathbb{Z}[X]^W$ and $\beta$, $\gamma$ in $X^+$ let $m(\xi)_{\beta,\gamma} = \mathrm{Mult}(s(\gamma)^\sigma st, s(\beta)\xi)$, and let $M(\xi)$ be the matrix indexed by $X^+$ whose $\beta - \gamma$ entry is $m(\xi)_{\beta,\gamma}$. Let $P_\rho$ be the permutation matrix indexed by $X^+$ whose $\beta - \gamma$ entry is 1 if $\beta^\rho = \gamma$ and 0 otherwise.

Let $\lambda \in X_\sigma$. Define matrices $A_\lambda^{(j)}$ with entries in $\mathbb{Z}[X]^W$ inductively as follows: Let

$$A_\lambda^{(1)} = \sum_{\substack{\mu \in X_\sigma \\ \mu \neq \lambda}} M(\bar{\psi}_\lambda \chi_p(\mu))\psi_\mu.$$

Assuming that $A_\lambda^{(j-1)}$ has been defined as a matrix indexed by $\prod_{i=1}^{j-1} X$, define $A_\lambda^{(j)}$ to be the matrix indexed by $\prod_{i=1}^{j} X$ whose $(\beta_1, ..., \beta_j) - (\gamma_1, ..., \gamma_j)$ component is

$$\sum_{\substack{\mu \in X_\sigma \\ \mu \neq \lambda}} \mathrm{Mult}(s(\gamma_j)^\sigma st, s(\beta_j)\bar{\xi}\chi_p(\mu))\psi_\mu,$$

where $\xi$ is the $(\beta_1, ..., \beta_{j-1}) - (\gamma_1, ..., \gamma_{j-1})$ entry of $A_\lambda^{(j-1)}$.

Let $P_\rho^{(j)}$ be the matrix indexed by $\prod_{i=1}^{j} X$ whose $\underline{\underline{\beta}} - \underline{\underline{\gamma}}$ entry is one if $\underline{\underline{\beta}}^\rho = \underline{\underline{\gamma}}$ and 0 otherwise.

THEOREM 1. $\Phi_{\lambda,\tau} = \sum_j \mathrm{Tr}(\prod_{i=1}^{n-1} (A_{\lambda(i)}^{(j)})^{\sigma^i} P_\rho^{(j)})$.

(We will prove that $A_\lambda^{(j)}$ is 0 for all but finitely many $j$.)

We now give a formula for the Cartan invariants.

For $\lambda$ and $\mu$ in $X_\tau$ let $C_{\lambda,\mu}^{(\tau)}$ be the number of times that $\chi_p(\mu)$ occurs in $\phi_{\lambda,\tau}$. If $i$ and $j$ are non-negative integers let $B_{\lambda,\mu}^{(i,j)}$ be the matrix indexed by $\prod_{k=1}^{i+j+1} X^+$ whose $(\beta_1, ..., \beta_{i+j+1}) - (\gamma_1, ..., \gamma_{i+j+1})$ entry is $\mathrm{Mult}(s(\gamma_{i+j+1})^\sigma st, s(\beta_{i+j+1}) \cdot \xi \cdot st)$ where $\xi$ is the product of the complex conjugate of the $(\beta_1, ..., \beta_i) - (\gamma_1, ..., \gamma_i)$ entry of $A_\lambda^{(i)}$ times the $(\beta_{i+1}, ..., \beta_j) - (\gamma_{i+1}, ..., \gamma_j)$ entry of $A_\mu^{(j)}$.

THEOREM 2. $C_{\lambda,\mu}^{(\tau)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \mathrm{Tr}(\prod_{k=0}^{n-1} B_{\lambda(k),\mu(k)}^{(i,j)})$.

## 3. Proof of the Theorems

By the orthogonality relations on $R(G_\tau)$ the number of times that $\phi_{\mu,\tau}$ occurs in $\psi_{\lambda,\tau}$ (considered as an element of $R(G_\tau)$) is equal to $(\Psi_{\lambda,\tau}, \chi_p(\mu))$, where the brackets denote the usual inner product on $R(G_\tau)$.

Thus,

$$\Psi_{\lambda,\tau} = \Phi_{\lambda,\tau} + \sum_{\mu \in X_\tau} (\Psi_{\lambda,\tau}, \chi_p(\mu))\Phi_{\mu,\tau}. \tag{1}$$

The idea now is to invert formula (1) to express the $\Phi$'s in terms of the $\Psi$'s. For $j \geqslant 0$ we define elements $\psi^{(j)}_{\lambda,\tau}$ of $P_\tau(G)$ inductively as follows.

Let $\psi^{(0)}_{\lambda,\tau} = \psi_{\lambda,\tau}$. Let $\psi^{(1)}_{\lambda,\tau} = \sum_{\mu \in X_\tau, \mu \neq \lambda} (\Psi_{\lambda,\tau}, \chi_p(\mu))\psi_{\mu,\tau}$, and assuming $\psi^{(j-1)}_{\lambda,\tau}$ has been defined, let $\psi^{(j)}_{\lambda,\tau} = \sum_{\mu \in X_\tau, \mu \neq \lambda} (\Psi^{(j-1)}_{\lambda,\tau}, \chi_p(\mu))\psi_{\mu,\tau}$, for $j \geqslant 1$.

Let $\Psi^{(j)}_{\lambda,\tau} = st_\tau \cdot \psi^{(j)}_{\lambda,\tau}$.

Lemma 1. $\Phi_{\lambda,\tau} = \sum_{j=0}^{\infty} (-1)^j \Psi^{(j)}_{\lambda,\tau}$.

*Proof.* This follows by inverting the formula (1). ∎

We will show below that for large enough $j$, $\Phi^{(j)}_{\lambda,\tau}$ is 0, so that the summation in Lemma 1 is in fact finite.

To find the projective indecomposables we must therefore evaluate the $\Psi^{(j)}_{\lambda,\tau}$, and to do this we need to compute certain inner products in $R(G_\tau)$. We shall do this by translating these inner products into multiplicities in $R(G_\tau)$ and $\mathbb{Z}[X]^W$. The following result shows that we can express the inner product of an element of $P(G_\tau)$ with one in $R(G_\tau)$ in terms of multiplicities in $R(G_\tau)$.

Lemma 2. *Let $\xi_1, \xi_2 \in R(G_\tau)$. Then $(St_\tau \cdot \xi_1, \xi_2) = \mathrm{Mult}_{G_\tau}(St_\tau, \bar{\xi}_1 \cdot \xi_2)$, where the multiplicity is with respect to the basis $\{\chi_p(\lambda) : \lambda \in X_\tau\}$ of $R(G_\tau)$.*

*Proof.* We have

$$(St_\tau \cdot \xi_1, \xi_2) = (St_\tau, \bar{\xi}_1 \cdot \xi_2).$$

Since $St_\tau$ is its own projective cover it follows from the orthogonality relations that this last inner product is equal to the number of times that $St_\tau$ occurs as an irreducible constituent of $\bar{\xi}_1, \xi_2$. This is precisely the multiplicity stated in the lemma. ∎

We now express the multiplicities occurring in Lemma 2 in terms of $\mathbb{Z}[X]^W$ multiplicities.

LEMMA 3. *Let* $\xi \in \mathbb{Z}[X]^W$. *Then* $\mathrm{Mult}_{G_\tau}(st_\tau, \xi) = \sum_{\lambda \in X^+} \mathrm{Mult}(s(\lambda)^\tau st_\tau, s(\lambda)\xi)$.

*Proof.* The proof of Lemma 8 of [1] is easily modified so that the result there holds for $\tau$ in place of the Frobenius map. The lemma is then just a special case of that result. ∎

The last 2 lemmas give the following formula for the inner product.

LEMMA 4. $(\Psi_{\lambda,\tau}, \chi_p(\mu)) = \sum_{\nu \in X^+} \mathrm{Mult}(s(\nu)^\tau st_\tau, s(\nu)\bar{\psi}_{\lambda,\tau}\chi_p(\mu))$.

We now use this result to show that the summation in Lemma 1 is finite.

LEMMA 5. *For every* $\lambda$ *in* $X^+$, $\Psi^{(j)}_{\lambda,\tau}$ *is* 0 *large enough* $j$.

*Proof.* Suppose $\Psi_{\gamma,\tau}$ appears in $\Psi^{(j)}_{\gamma,\tau}$ and $\Psi_{\eta,\tau}$ appears in $\Psi^{(j+1)}_{\lambda,\tau}$ where $\gamma$ and $\eta$ are in $X^+$. Then $(\Psi_{\gamma,\tau}, \chi_p(\eta))$ is non-zero so $\mathrm{Mult}(s(\nu)^\tau st_\tau, s(\nu)\bar{\psi}_{\gamma,\tau}\chi_p(\eta))$ is non-zero for some $\nu$ in $X^+$. Moreover $\nu \neq 0$ since $\eta \neq \gamma$. The highest weight of $s(\nu)^\tau st_\tau$ is $\nu^\tau + \delta^\tau - \delta$ while that of $s(\nu)\bar{\psi}_{\gamma,\tau}\chi_p(\eta)$ is $\nu + \delta^\tau - \delta - \gamma + \eta$. Thus $\nu^\tau + \delta^\tau - \delta \leqslant \nu + \delta^\tau - \delta - \gamma + \eta$, or $\eta \geqslant \nu^\tau - \nu + \gamma$, where the ordering is the usual partial one on $X^+$. Thus $\langle \eta, \alpha_0^v \rangle > \langle \gamma, \alpha_0^v \rangle$ where $\alpha_0^v$ is the dual to the highest short root (i.e., the highest long co-root) in the root system of $G$. But since $\eta \in X_\tau$, we also have $\langle \eta, \alpha_0^v \rangle \leqslant \langle \delta^\tau - \delta, \alpha_0^v \rangle$. There can thus be only a finite number of non-zero $\Psi^{(j)}_{\lambda,\tau}$, which proves the lemma. ∎

We now express the sum of multiplicities given in Lemma 3 as the trace of a product of matrices.

LEMMA 6. *Let* $\xi_i$, $i = 0, ..., n-1$, *be elements of* $\mathbb{Z}[X]^W$. *Then* $\sum_{\lambda \in X^+} \mathrm{Mult}(s(\lambda)^\tau st_\tau, s(\lambda) \prod_{i=0}^{n-1} \xi_i^{\sigma^i}) = \mathrm{Tr} \prod_{i=0}^{n-1} M(\xi_i) P_\rho$.

*Proof.* The sum of multiplicities is equal to $\sum \prod_{i=0}^{n-1} \mathrm{Mult}(s(\lambda_{i+1})^\sigma st, s(\lambda_i)\xi)$ the sum being over all $\lambda_1, ..., \lambda_n$ in $X^+$ such that $\lambda_n^\rho = \lambda_0$. But this is precisely the trace of the product of matrices stated. ∎

We now prove our main results

*Proof of Theorem* 1. We first show that

$$\psi^{(j)}_{\lambda,\tau} = \mathrm{Tr}\left(\prod_{i=0}^{n-1} (A^{(j)}_{\lambda(i)})^{\sigma^i} P^{(j)}_\rho\right). \tag{2}$$

For $j = 1$ this follows from Lemma 2 by applying Lemma 6 to $\xi = \sum_{\mu \in X_\tau} \bar{\psi}_\lambda \chi_p(\mu)$. Assuming the formula for $j-1$,

$$\begin{aligned}\psi_{\lambda,\tau}^{(j)} &= \sum_{\substack{\mu\in X_\tau\\ \mu\neq\lambda}} (\psi_{\lambda,\tau}^{(j-1)}, \chi_p(\mu))\psi_{\mu,\tau}\\ &= \sum_{\substack{\mu\in X_\tau\\ \mu\neq\lambda}} (st_\tau, \bar{\psi}_{\lambda,\tau}^{(j-1)}\chi_p(\mu))\psi_{\mu,\tau}\\ &= \sum_{\substack{\mu\in X_\tau\\ \mu\neq\lambda}} \sum_{\gamma\in X^+} \mathrm{Mult}(s(\gamma)^\tau st_\tau, s(\gamma)\bar{\psi}_{\lambda,\tau}^{(j-1)}\chi_p(\mu))\psi_{\mu\tau}.\end{aligned}$$

Using the formula for $\psi_{\lambda,\tau}^{(j-1)}$ and an argument similar to the one in Lemma 6, gives this sum as the trace of the matrix stated.

The proof of the theorem now follows from the formula in Lemma 1. ∎

*Proof of Theorem* 2. It follows from the definitions, Lemmas 2, 3, and 6, and Theorem 1 that

$$(\psi_\lambda^{(i)}, \psi_\lambda^{(j)}) = \mathrm{Tr}\left(\prod_{k=0}^{n-1} B_{\lambda(k),\mu(k)}^{(i,j)}\right).$$

On the other hand, by the orthogonality relations in $R[G_\tau]$, $C_{\lambda,\tau}^{(\tau)} = (\Phi_{\lambda,\tau}, \Phi_{\mu,\tau})$. The formula now follows from Lemma 1. ∎

We conclude by proving a lemma which makes it easier to calculate inner products of elements in $P_\tau(G)$. First recall the following well-known formula (see [4,.Exercise 24.9]): If $\lambda, \mu \in X$ then

$$s(\lambda)\chi(\mu) = \sum_{\lambda'\sim\lambda} \chi(\mu+\lambda') \tag{3}$$

the sum being over all $W$-conjugates of $\lambda$.

Also note that

$$\chi(\lambda)^\sigma st = \chi((\lambda+\delta)^\sigma - \delta) \tag{4}$$

since by definition the left hand side is equal to

$$\left(\sum_{w\in W} (\mathrm{sgn}\, w)e(w(\lambda+\delta)^\sigma)\Big/\sum_{w\in W} (\mathrm{sgn}\, w)e(w\delta^\sigma)\right)$$
$$\cdot\left(\sum_{w\in W} (\mathrm{sgn}\, w)e(w\delta^\sigma)\Big/\sum_{w\in W} (\mathrm{sgn}\, w)e(w\delta)\right),$$

which is equal to the right hand side.

LEMMA 7. *Let* $\xi\in\mathbb{Z}[X]^W$. *Then* $\mathrm{Mult}(s(\lambda)^\sigma st, \xi\cdot st) = \mathrm{mult}(s(\lambda^\sigma), \xi)$, *where the second multiplicity is with respect to the basis* $\{s(\lambda) : \lambda\in X^+\}$ *of* $\mathbb{Z}[X]^W$.

*Proof.* The lemma will follow if we show that $\mathrm{Mult}(s(\lambda)^\sigma st, s(\mu)\cdot st)=0$ for all $\lambda\in X$ unless $\mu\in X^\sigma=\{\lambda^\sigma:\lambda\in X\}$. Now if this multiplicity is non-zero for some $\lambda$ then $\mathrm{Mult}(\chi(\lambda)^\sigma st, s(\mu)\cdot st)$ is non-zero for some $\lambda\in X$. Then by (3) and (4), $w\mu+\delta^\sigma-\delta=(\lambda+\delta)^\sigma-\delta$ for some $w\in W$, and so $w\mu\in X^\sigma$ as required. ∎

This lemma in particular makes the computation of the multiplicities appearing in the definition of the matrix $B^{(i,j)}_{\lambda,\mu}$ more efficient.

## 4. An Example: $SL_3(q)$ and $SU_3(q)$

We illustrate the ideas in the previous section by examining the projective cover of the trivial module and the first Cartan invariant for $SL_3(p^n)$ and $SU_3(p^n)$ for $p\geqslant 5$.

We now suppose that $G$ is one of $SL_3(K)$ or $SU_3(K)$ and that $p\geqslant 3$. Let $\sigma$ be the Frobenius map and let $\rho$ be either the identity (to get $SL_3(p^n)$) or an automorphism of order 2 (to get $SU_3(p^n)$).

We shall denote elements of $X$ by their coordinates in terms of the fundamental dominant weights. The decomposition of $\chi(\lambda)$ for the cases under consideration is well known. If $\lambda=(a,b)$ is in $X_\sigma$ then $\chi(\lambda)=\chi_p(\lambda)$ if $a+b\leqslant p-2$, while $\chi(\lambda)=\chi_p(\lambda)+x_p(\lambda')$ where $\lambda'=(p-b-2, p-a-2)$ if $a+b>p-2$. It follows from [1, Lemma 2] that $\Psi_\lambda=st\cdot s(p-1-b, p-1-a)$ if $a+b\geqslant p-1$ while $\Psi_\lambda=st\cdot(s(p-1-b, p-1-a)+s(a+1, b+1)$ if $a+b<p-1$. In particular, $\Psi_{(0,0)}=st\cdot(s(1,1)+s(p-1,p-1))$.

To calculate the traces of products of the matrices $M(\xi)$ it will be convenient to introduce the following partial order on $X^+$: $\lambda\leqslant\mu$ will mean that either $\mu-\lambda$ is a sum of positive roots, or each coordinate of $\lambda$ in terms of the fundamental weights is at most equal to the corresponding coordinate of $\mu$. It is clear that this is indeed a partial order.

Abbreviate $\lambda=(0,0)$ by 0. To calculate $\Phi_0$ by the formula in Section 2 we need to know when $M(\psi_0\chi_p(\mu))$ can be non-zero.

Lemma 8. *If $\mu$ is a non-zero restricted dominant weight, then the $\beta,\gamma$ entry of $M(\psi_0\chi_p(\mu))$ is 0 unless $\gamma\lneqq\beta$ except for the following cases, in each of which the entry is one:*

(1) $\mu=(p-1,0)$, $\beta=\gamma=(1,0)$.

(2) $\mu(0,p-1)$, $\beta=\gamma=(0,1)$.

(3) $\mu=(p-1,1)$, $\beta=(0,1)$, $\gamma=(1,0)$.

(4) $\mu(1,p-1)$, $\beta=(1,0)$, $\gamma=(0,1)$.

(5) $\mu(p-1,p-1)$, $\beta=\gamma=(1,1)$.

*Proof.* If $\mu=(a, b)$ then $\psi_0\chi(\mu)$ can be written as a sum of $\chi(\nu)$ using formula (3) of the previous section. It is easy to check that the $u-\gamma$ entry of $M(\chi(\nu))$ for each such term is 0 unless $\gamma \lneqq \beta$ except when $\nu=(p-1+a,\ p-1+b)$. The value of the entry is then one precisely for the values of $\mu$, $\beta$, and $\gamma$ stated in the lemma. In each case $a+b>p-2$, so when $\lambda$ is in $X_\sigma$, $\chi_p(\lambda)$ can have such a $\chi(\mu)$ appearing with non-zero multiplicity only when $\lambda=\mu$. The lemma now follows. ∎

We now use this lemma and formula (2) to compute $\psi^{(1)}_{0,\tau}$.

LEMMA 9.

$$\psi^{(1)}_{0,\tau}=\mathrm{Tr}\left(\prod_{i=0}^{n-1}\begin{bmatrix}\psi_{(p-1,0)} & \psi_{(1,p-1)} & 0\\ \psi_{(p-1,1)} & \psi_{(0,p-1)} & 0\\ 0 & 0 & 1\end{bmatrix}P_\rho\right),$$

*where $P_\rho$ is the $3\times 3$ identity matrix if $\rho$ is the identity and is equal to*

$$\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

*otherwise.*

*Proof.* By Lemma 8 the only part of $A^{(1)}_0$ which will contribute to $\psi^{(1)}_{0,\tau}$ as given by the trace formula in (2) is the part indexed by (1, 0), (0, 1), and (1, 1), and this part of $A^{(1)}_0$ is precisely the matrix displayed in the lemma. (Note that $\psi_{(p-1,p-1)}=1$.) ∎

It will be a bit more convenient if we split off a $st_\tau$ from $\psi^{(1)}_{0,\tau}$. Accordingly, define $\psi'_{0,\tau}=\psi^{(1)}_{0,\tau}-1$ and let $\Psi'_{0,\tau}=St_\tau\cdot\psi'_{0,\tau}$. Thus by Lemma 9 we have

$$\psi'_{0,\tau}=\mathrm{Tr}\left(\prod_{i=0}^{n-1}\begin{pmatrix}\psi_{(p-1,0)} & \psi_{(1,p-1)}\\ \psi_{(p-1,1)} & \psi_{(0,p-1)}\end{pmatrix}^{\sigma^i}P_\rho\right), \tag{5}$$

where $P_\rho$ is the identity or $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ according as to whether $\rho$ is or is not the identity.

To find $\psi^{(2)}_{0,\tau}$ we must now compute the matrices $M(\bar{\psi}_\lambda\chi_p(u))$ where $\psi_\lambda$ is a term appearing in $A^{(1)}_0$ as in Lemma 9.

LEMMA 10. *Suppose $\lambda$ either $(p-1, 0)$, $(0, p-1)$, $(p-1, 1)$, $(1, p-1)$, or $(p-1,\ p-1)$ and $\mu$ is in $X_p$. The $\beta-\gamma$ entry of $M(\bar{\psi}_\lambda\chi_p(u))$ is 0 unless $\gamma\lneqq\beta$ except in the following cases when the entry is one:*

(1) $\lambda=(p-1,0)$, $\mu=(p-1,p-1)$, $\beta=\gamma=(0,1)$.
(2) $\lambda=(0,p-1)$, $\mu=(p-1,p-1)$, $\beta=\gamma=(1,0)$.
(3) $\lambda=(p-1,p-1)$, $\mu=(p-1,p-1)$, $\beta=\gamma=(1,1)$.

*Proof.* We have

$$\begin{aligned}
\overline{\psi(p-1,0)}&=\overline{s(p-1,0)}=s(0,p-1)\\
\overline{\psi(0,p-1)}&=\overline{s(0,p-1)}=s(p-1,0)\\
\overline{\psi(p-1,1)}&=\overline{s(p-2,0)}=s(0,p-2)\\
\overline{\psi(1,p-1)}&=\overline{s(0,p-2)}=s(p-2,0)\\
\overline{\psi(p-1,p-1)}&=s(0,0).
\end{aligned}$$

We can use (3) to write each $\bar{\psi}_\lambda\chi(\mu)$ as a sum of $\chi(\nu)$. Examining each term it is easy to check that the $\beta-\gamma$ entry of $M(\chi(\nu))$ for each such $\chi(\nu)$ is 0 unless $\gamma\lneqq\beta$ except when $\nu=\lambda+\mu$ when the entry is one for the values of $\lambda$, $\mu$, $\beta$, and $\gamma$ stated in the lemma. In each case $\chi(\mu)=St$, which does not appear with non-zero multiplicity for any $\chi_p(\mu)$ when $\mu$ is restricted except for $\mu=(p-1)\delta$. The lemma follows. ∎

LEMMA 11.

$$\begin{aligned}
\psi^{(2)}_{0,\tau}&=2 \qquad \textit{if} \quad \rho \textit{ is the identity}\\
&=0 \qquad \textit{otherwise.}
\end{aligned}$$

*Proof.* By Lemma 10 the only part of $A_0^{(2)}$ which contributes to $\psi_0^{(\tau)}$ as given by the trace formula (2) is the part indexed by ((0, 1), (1, 0)) and ((1, 0), (0, 1)) and this part is equal to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. The corresponding part of $P_\rho$ is the identity if $\rho=1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ otherwise. The lemma then follows from Theorem 1. ∎

LEMMA 12. $\psi^{(j)}_{0,\tau}=0$ *for* $j\geqslant 3$.

*Proof.* $\Psi^{(2)}_{0,\tau}$ is either $2St_\tau$ or 0 and $(St_\tau,\chi_p(\mu))$ is 0 for $\mu$ in $X_\tau$ unless $\mu=(p-1)\delta$. ∎

Putting these results together we have the following formula for $\Phi_{0,\tau}$.

THEOREM 3. $\Phi_{0,\tau}=\Psi_{0,\tau}-\Psi'_{0,\tau}+St_\tau$ *if* $\rho=1$ *and* $\Phi_{0,\tau}=\Psi_{0,\tau}-\Psi'_{0,\tau}-St_\tau$ *if* $\rho\neq 1$, *where* $\Psi'_{0,\tau}=St_\tau\cdot\psi'_{0,\tau}$ *is given by* (5).

*Proof.* This follows from Theorem 1 and Lemmas 9, 11, and 12.

COROLLARY. *The degree of* $\Phi_{0,\tau}$ *is equal to* $p^{3n}(12^n - 6^n + 1)$ *if* $\rho = 1$ *and is equal to* $p^{3n}(12^n - 6^n - 1)$ *if* $\rho \neq 1$.

*Proof.* This follows by evaluating the formula in Theorem 3 at 1. ∎

We now proceed to the determination of the first Cartan invariant $C_{0,0}^{(\tau)} = (\Phi_{0,\tau}, \Phi_{0,\tau})$. Since $St_\tau$ is also an irreducible $G_\tau$ module we have $(\Phi_{0,\tau}, St_\tau) = 0$. Thus $(\Phi_{0,\tau} \pm St_\tau, \Phi_{0,\tau} \pm St_\tau) = C_{0,0}^{(\tau)} + 1$. Thus by Theorem 3 we obtain

$$C_{0,0}^{(\tau)} = (\Psi_{0,\tau} - \Psi'_{0,\tau}, \Psi_{0,\tau} - \Psi'_{0,\tau}) - 1. \tag{3.9}$$

The calculation now proceeds by considering separately each of the terms $(\Psi_{0,\tau}, \Psi_{0,\tau})$, $(\Psi_{0,\tau}, \Psi'_{0,\tau})$, and $(\Psi'_{0,\tau}, \Psi'_{0,\tau})$.

LEMMA 13. *Suppose* $p \geqslant 7$. *Then* $(\Psi_{0,\tau}, \Psi_{0,\tau}) = 1 + 2 \cdot 2^n + a^n + b^n$ *if* $\rho = 1$ *and is equal to* $1 + a^n + b^n$ *if* $\rho \neq 1$, *where* $a$ *and* $b$ *are distinct roots of the polynomial* $x^2 - 18x + 48$.

*Proof.* By the formula in Section 3, $(\Psi_{0,\tau}, \Psi_{0,\tau})$ is equal to the trace of $(M(\psi_0^2 St))^n P_\rho$. Now Lemma 7 can be used to somewhat simplify the calculation of the entries of this matrix. By that result the $\beta - \gamma$ entry is equal to $\mathrm{Mult}(s(p\gamma), \psi_0^2 s(\beta))$. Now $\psi_0^2 = (s(p-1, p-1) + s(1, 1))^2$ can be expanded as a linear combination of $s(v)$. The $\beta - \gamma$ entry of the matrix corresponding to each such term is 0 unless $\gamma \lneqq \beta$ except for the following cases:

(1) $v = (2p-2, 2p-2)$, $\beta = \gamma = (2, 2)$
(2) $v = (0, 3p-3)$, $\beta = \gamma = (0, 3)$
(3) $v = (3p-3, 0)$, $\beta = \gamma = (3, 0)$
(4) $v = (p-1, p-1)$, $(p-2, p+1)$, or $(p+1, p-2)$, $\beta = \gamma = (1, 1)$
(5) $v = (0, 0)$, $\beta = \gamma = (0, 0)$
(6) $v = (p, p)$, $\beta = (0, 0)$, $\gamma = (1, 1)$.

Because of (6) the $(1, 1) - (0, 0)$ entry of $M(\psi_0^2 st)$ also contributes to the trace we are looking for. Thus we must also consider

(7) $v = (1, 1)$, $\beta = (1, 0)$, $\gamma = (0, 0)$.

The entry in each case is one except for (7) where it is 6. The coefficient of $s(v)$ in $\psi_0^2$ is 1 in case (1), it is 2 in cases (2)-(4), (6), and (7), and it is 12 in case (5).

Thus the only part of $M(\psi_0^2 st)$ that wil contribute to the trace we are

computing is indexed by (2, 2), (0, 3), (3, 0), (1, 1), and (0, 0). This part is equal to

$$\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 12 & 2 \\ & & & 12 & 6 \end{bmatrix}.$$

When $\rho = 1$, $(\Psi_{0,\tau}, \Psi_{0,\tau})$ is then equal to the trace of the $n$th power of the matrix, which is given by the statement of the lemma. If $\rho \neq 1$ then the inner product is equal to $1 + \mathrm{Tr}\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)^n \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \mathrm{Tr}\left(\begin{smallmatrix} 12 & 2 \\ 12 & 6 \end{smallmatrix}\right)^n$ which is again as in the statement of the lemma. ∎

LEMMA 14. *Suppose $p = 5$. Then*

$$\begin{aligned} (\Psi_{0,\tau}, \Psi_{0,\tau}) &= 1 + 3 \cdot 2^n + (-2)^n + a^n + b^n \qquad \textit{if} \quad \rho = 1 \\ &= 1 + 2^n - (-2)^n + a^n + b^n \qquad \textit{if} \quad \rho \neq 1, \end{aligned}$$

*where $a$ and $b$ are as in the previous lemma.*

*Proof.* We have the same argument as in Lemma 13 except that there are 2 additional cases for $s(v)$ to be considered:

(8) $v(3p-3, 0)$, $\beta = (0, 2)$, $\gamma = (2, 0)$

(9) $v = (0, 3p-3)$, $\beta = (2, 0)$, $\gamma = (0, 2)$.

The entry is one and $s(v)$ occurs in $\psi_0^2$ with coefficient 2 in both cases. Thus there is an additional contribution to the inner product of $\mathrm{Tr}\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)^n$ if $\rho = 1$ and of $\mathrm{Tr}\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)^n \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^n$ if $\rho \neq 1$. This gives the modified expression for $(\Psi_{0,\tau}, \Psi_{0,\tau})$. ∎

LEMMA 15. *Suppose $p \geqslant 5$. Then*

$$\begin{aligned} (\Psi_0, \Psi_0') &= 2 + 2 \cdot 2^n + 8^n \qquad \textit{if} \quad \rho = 1 \\ &= 8^n \qquad \textit{if} \quad \rho \neq 1. \end{aligned}$$

*Proof.* Using the definition of $\psi_0'$ we see that the inner product is equal to the trace of the matrix $B^n P_\rho^{(2)}$ where $B$ is the matrix indexed by $X^+ \times X^+$ whose $(\beta_1, \beta_2) - (\gamma_1, \gamma_2)$ entry is $\sum_{\mu \neq 0, \mu \neq (p-1)\delta} m(\psi_0 \chi_p(\mu))_{\beta_1, \gamma_1} m(\psi_0 \Psi_\mu)_{\beta_2, \gamma_2}$.

By Lemma 8 the only $\mu$ we need consider are $\mu = (p-1, 0)$, $(0, p-1)$, $(p-1, 1)$, or $(1, p-1)$. We compute $M(\psi_0 \Psi_\mu)$ by using Lemma 7. Multiplying out $\psi_0 \psi_\mu$ as a linear combination of $s(v)$ for the above values of $\mu$

and computing $\mathrm{Mult}(s(p\gamma), s(\beta)s(\nu))$ we see that the values are 0 unless $\gamma \lneqq \beta$ except for the following cases:

(1) $\mu=(p-1,0)$, $\nu=(p-1,2p-2)$, $\beta=\gamma=(1,2)$.

(2) $\mu=(0,p-1)$, $\nu=(2p-2,p-1)$, $\beta=\gamma=(2,1)$.

(3) $\mu=(p-1,0)$, $\nu(2p-2,0)$, $\beta=\gamma=(2,0)$.

(4) $\mu=(0,p-1)$, $\nu=(0,2p-2)$, $\beta=\gamma=(0,2)$.

(5) $\mu=(p-1,0)$, $\nu=(0,p-1)$, $\beta=\gamma=(0,1)$.

(6) $\mu=(0,p-1)$, $\nu=(p-1,0)$, $\beta=\gamma=(0,1)$.

(7) $\mu=(p-1,0)$, $\nu=(1,p)$, $\beta=\gamma=(0,1)$.

(8) $\mu=(0,p-1)$, $\nu(p,1)$, $\beta=\gamma=(1,0)$.

(9) $\mu=(1,p-1)$, $\nu=(p-1,1)$, $\beta=(0,1)$, $\gamma=(1,0)$

(10) $\mu=(p-1,1)$, $\nu=(1,p-1)$, $\beta=(1,0)$, $\gamma=(0,1)$.

The multiplicity of $s(p\gamma)$ in cases (1)–(6) is one and it is two in cases (7)–(10). The coefficient of $s(\nu)$ is one in cases (1), (2), (7), and (8) and it is two in the others. Thus the only part of $B$ contributing to the trace of $B^nP_\rho^{(2)}$ is indexed by $((1,0),(1,1))$, $((0,1),(2,1))$, $((1,0),(2,0))$, $((0,1),(0,2))$, $((1,0),(0,1))$, and $((0,1),(1,0))$ and is equal to

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 4 & 4 \\ & & & & 4 & 4 \end{bmatrix}.$$

Computing $B^nP_\rho^{(2)}$ gives the formula for its trace as stated in the lemma. ∎

LEMMA 16. *Assume* $p \geqslant 5$.

$$(\Psi_0', \Psi_0') = 4 + 2\cdot 2^n + 6^n.$$

*Proof.* The inner product is equal to the trace of $C^nP_\rho^{(3)}$ where $C$ is the matrix indexed by $X^+\times X^+\times X^+$ whose $(\beta_1,\beta_2,\beta_3)-(\gamma_1,\gamma_2,\gamma_3)$ entry is

$$\sum_{\substack{\lambda,\mu\neq 0\\ \lambda,\mu\neq(p-1)\rho}} m(\psi_0\chi_p(\mu))_{\beta_1,\gamma_1} m(\psi_0\chi_p(\nu))_{\beta_2,\gamma_2} m(\bar{\psi}_\mu\Psi_\lambda)_{\beta_3,\gamma_3}.$$

By Lemma 10, the only $\mu$ or $\lambda$ we need consider are either $(p-1,0)$, $(0,p-1)$, $(p-1,1)$, or $(1,p-1)$. Again we use Lemma 7. Multiplying out the various $\bar{\psi}_\mu\psi_\lambda$ as linear combinations of $s(\nu)$ for the values of $\mu$ and $\lambda$ given above and computing $\mathrm{Mult}(s(p\gamma), s(\beta)s(\nu))$, we see that the values are 0 unless $\gamma \lneqq \beta$ except for the following cases:

(1) $\mu=\lambda=(p-1,0)$, $\nu=(2p-2,0)$, $\beta=\gamma=(2,0)$.
(2) $\mu=\lambda=(0,p-1)$, $\nu=(0,2p-2)$, $\beta=\gamma=(0,2)$.
(3) $\mu=(p-1,0)$, $\lambda=(0,p-1)$, $\nu=(0,p-1)$, $\beta=\gamma=(0,1)$.
(4) $\mu=(0,p-1)$, $\lambda=(p-1,0)$, $\nu=(p-1,0)$, $\beta=\gamma=(1,0)$.
(5) $\mu=\lambda=(p-1,0)$, $\nu=(p-1,p-1)$, $\beta=\gamma=(1,1)$.
(6) $\mu=\lambda=(0,p-1)$, $\nu=(p-1,p-1)$, $\beta=\gamma=(1,1)$.
(7) $\mu=\gamma=(p-1,0)$, $\nu=(0,0)$, $\beta=\gamma=(0,0)$.
(8) $\mu=\lambda=(0,p-1)$, $\nu=(0,0)$, $\beta=\gamma=(0,0)$.
(9) $\mu=\gamma=(1,p-1)$, $\nu=(0,0)$, $\beta=\gamma=(0,0)$.
(10) $\mu=\gamma=(p-1,1)$, $\nu=(0,0)$, $\beta=\gamma(0,0)$.

The multiplicity of $s(p\gamma)$ in each case is one. The coefficient of $s(\nu)$ is one in cases (1), (2), (5), and (6); it is 2 in cases (3) and (4); and it is 3 in cases (7)–(10). Thus the only part of $C$ which will contribute to $\mathrm{Tr}(C^n P_\rho^{(3)})$ is indexed by

$$((1,0),(0,1),(2,0))$$
$$((0,1),(1,0),(0,2)),$$
$$((1,0),(0,1),(0,1)),$$
$$((0,1),(1,0),(1,0)),$$
$$((0,1),(0,1),(1,1)),$$
$$((1,0),(1,0),(1,1)),$$
$$((0,1),(0,1),(0,0)),$$

and

$$((1,0),(1,0),(0,0)).$$

This part of $C$ is equal to

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 3 & 3 \\ & & & & & & 3 & 3 \end{bmatrix}.$$

Computing $C^n P_\rho^{(3)}$ gives the formula for its trace as stated in the lemma. ∎

THEOREM 4. *If $p \geqslant 7$ then $C^{\tau}_{0,0} = a^n + b^n + 6^n - 2 \cdot 8^n$, where $a$ and $b$ are as in Lemma* 13. *If $p = 5$ then*

$$\begin{aligned} C^{\tau}_{0,0} &= a^n + b^n + 6^n - 2 \cdot 8^n + 2^n + (-2)^n \qquad \text{if} \quad \rho = 1 \\ &= a^n + b^n + 6^n - 2 \cdot 8^n + 2^n - (-2)^n \qquad \text{if} \quad \rho \neq 1. \end{aligned}$$

*Proof.* This follows from formula (6) using Lemmas 13–16. ∎

*Remark.* The cases $p = 2$ and $p = 3$ can also be handled by the above method. (Cheng [6] has done the case $p = 2$.) The computation for $p = 2$ differs from other $p$ because the formula for $\Phi_0$ is different. The case $p = 3$ is similar to the case for general $p$ but there are additional $s(v)$ to be considered in Lemmas 13, 15, and 16, which makes the resulting matrices more complicated.

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